# LECTURE 02

## 21CSE451T

# FUZZY LOGIC AND ITS APPLICATIONS

### Unit 01: Introduction




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# SYLLABUS

| | |
|---|---|
| **Unit-1 - Introduction** | **9 Hour** |
| The Case for Imprecision, The Utility of Fuzzy Systems, Limitations of Fuzzy Systems, Uncertainty and Information, Fuzzy Sets and Membership, Chance versus Fuzziness - Fuzzy Sets: Fuzzy Set Operations, Properties of Fuzzy Sets, Noninteractive Fuzzy Sets, Alternative Fuzzy Set Operations - Fuzzy Relations: Crisp Relations, Fuzzy Relations, Fuzzy Tolerance and Equivalence Relations, Value Assignments, Problems on fuzzy relation - Membership function – various forms –fuzzification – defuzzification to crisp sets. | |
| **Unit-2 - Logic and Fuzzy Systems** | **9 Hour** |
| classical logic, fuzzy logic, fuzzy systems – Development of Membership functions: membership value assignments, intuition, Inference, rank ordering – Automated Methods for Fuzzy Systems: Definitions, Batch Least Squares Algorithm, Recursive Least Squares Algorithm, Gradient Method, Learning From Example, Modified Learning From Example, Problems on logic and fuzzy systems | |
| **Unit-3 - Rule-Base Reduction Methods** | **9 Hour** |
| : Fuzzy Systems Theory and Rule Reduction, Singular Value Decomposition, Combs Method, SVD and Combs Method Examples, problems on SVD and Combs method for rapid inference - Decision Making with Fuzzy Information: Fuzzy Synthetic Evaluation, Fuzzy Ordering, Nontransitive Ranking, Preference and Consensus, Multiobjective Decision Making, Decision Making under Fuzzy States and Fuzzy Actions, problems on decision making with fuzzy information. | |
| **Unit-4 - Fuzzy Classification and Pattern Recognition** | **9 Hour** |
| Classification by Equivalence Relations, Cluster Analysis, Cluster Validity, c-Means Clustering, Fuzzy c-Means, Classification Metric, Similarity Relations from Clustering - Pattern Recognition: Feature Analysis, Partitions of the Feature Space, Single-Sample Identification, Multifeature Pattern Recognition, problems on fuzzy classification and pattern recognition, Case Study: Hand written character recognition using fuzzy logic. | |
| **Unit-5 - Fuzzy Control Systems** | **9 Hour** |
| Control System Design Problem, Control (Decision) Surface, Assumptions in a Fuzzy Control System Design, Simple Fuzzy Logic Controllers, Examples of Fuzzy Control System Design, Aircraft Landing Control Problem - Fuzzy Optimization - Fuzzy Linear Regression – problems on fuzzy optimization and regression, Case study: Robot Navigation using fuzzy logic. | |

# LECTURE 02 TOPICS

| | |
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| | |
|---|---|
| 1.7 | Fuzzy Set Operations |
| 1.8 | Properties of Fuzzy Sets |
| 1.9 | Noninteractive Fuzzy Sets |
| 1.1 | Alternative Fuzzy Set Operations |
| 1.11 | Crisp Relations |
| 1.12 | Fuzzy Relations |

# 1.7.1 Fuzzy Set Operations

Fuzzy set operations are extensions of classical set operations.

**Common fuzzy operations are:**

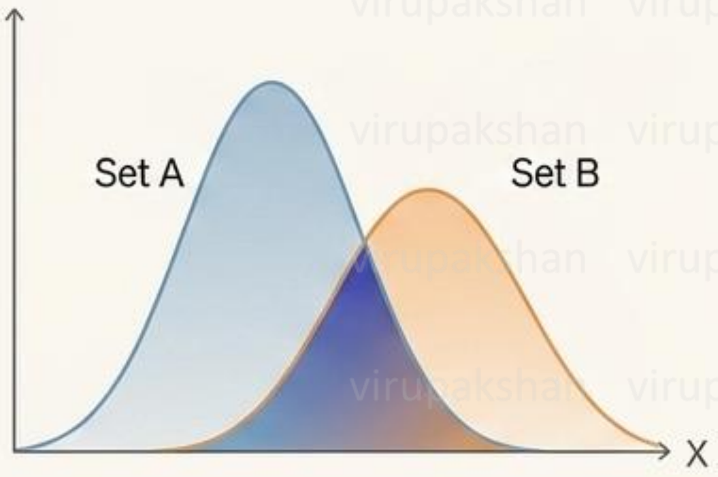
1. Intersection ( $A \cap B$ ) $$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$
2. Union (A U B) $$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$
3. Complement ( $A'$ ) $$\mu_{A'}(x) = 1 - \mu_A(x)$$

```
A intersection B → take smaller membership
A union B        → take larger membership
A complement     → subtract from 1
```

# 1.7.2Fuzzy Set Operations



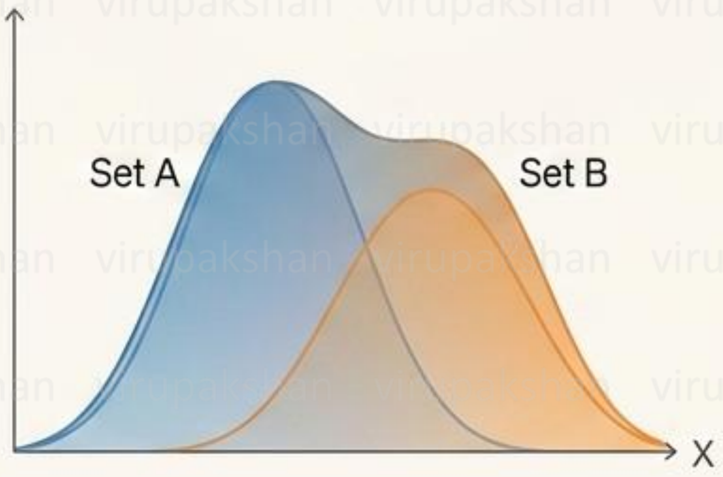
### Intersection (AND)




$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Min(0.8, 0.4) = 0.4

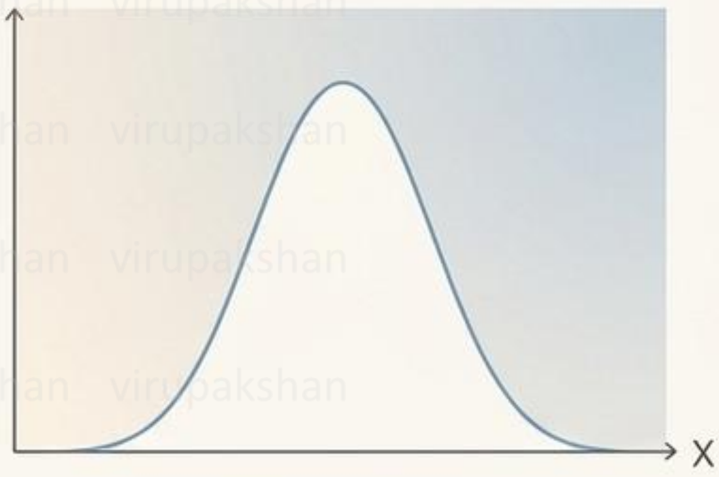
### Union (OR)




$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Max(0.8, 0.4) = 0.8

### Complement (NOT)

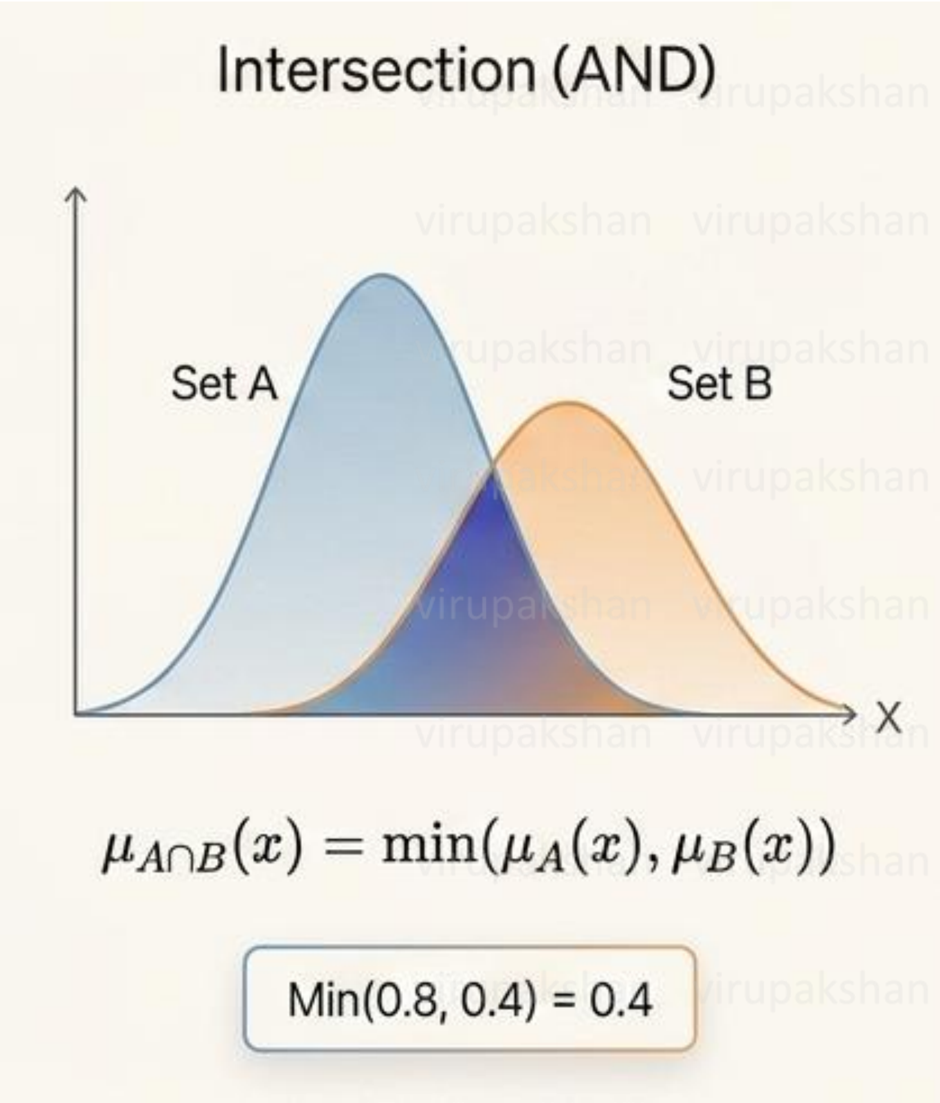



$$\mu'_{A'}(x) = 1 - \mu_A(x)$$

1 − 0.8 = 0.2

# 1.7.2a Fuzzy Set Operations: **Intersection** ( $A \cap B$ )

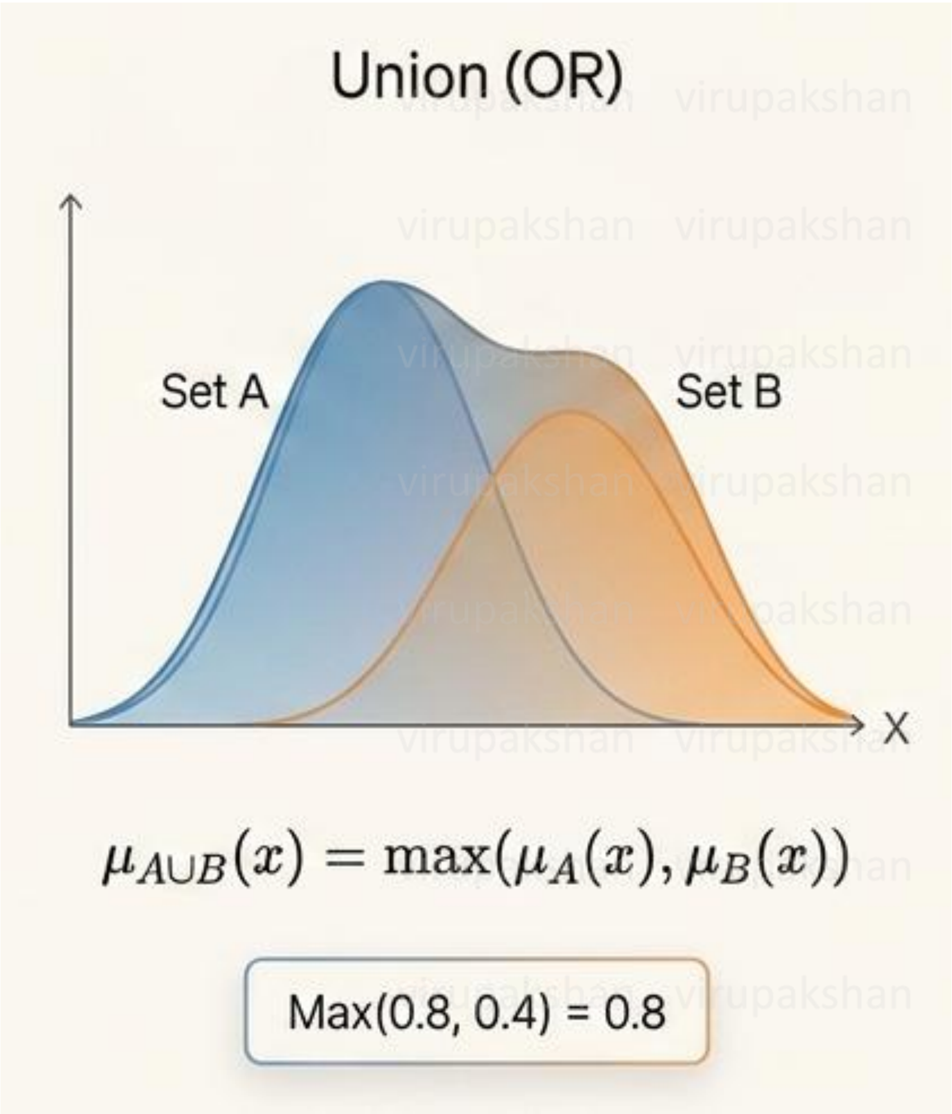





| **x** | $\mu_A(x)$ | $\mu_B(x)$ |
|---|---|---|
| x1 | 0.2 | 0.6 |
| x2 | 0.7 | 0.4 |
| x3 | 1.0 | 0.8 |

| **x** | **Intersection** |
|---|---|
| x1 | min(0.2, 0.6) = 0.2 |
| x2 | min(0.7, 0.4) = 0.4 |
| x3 | min(1.0, 0.8) = 0.8 |

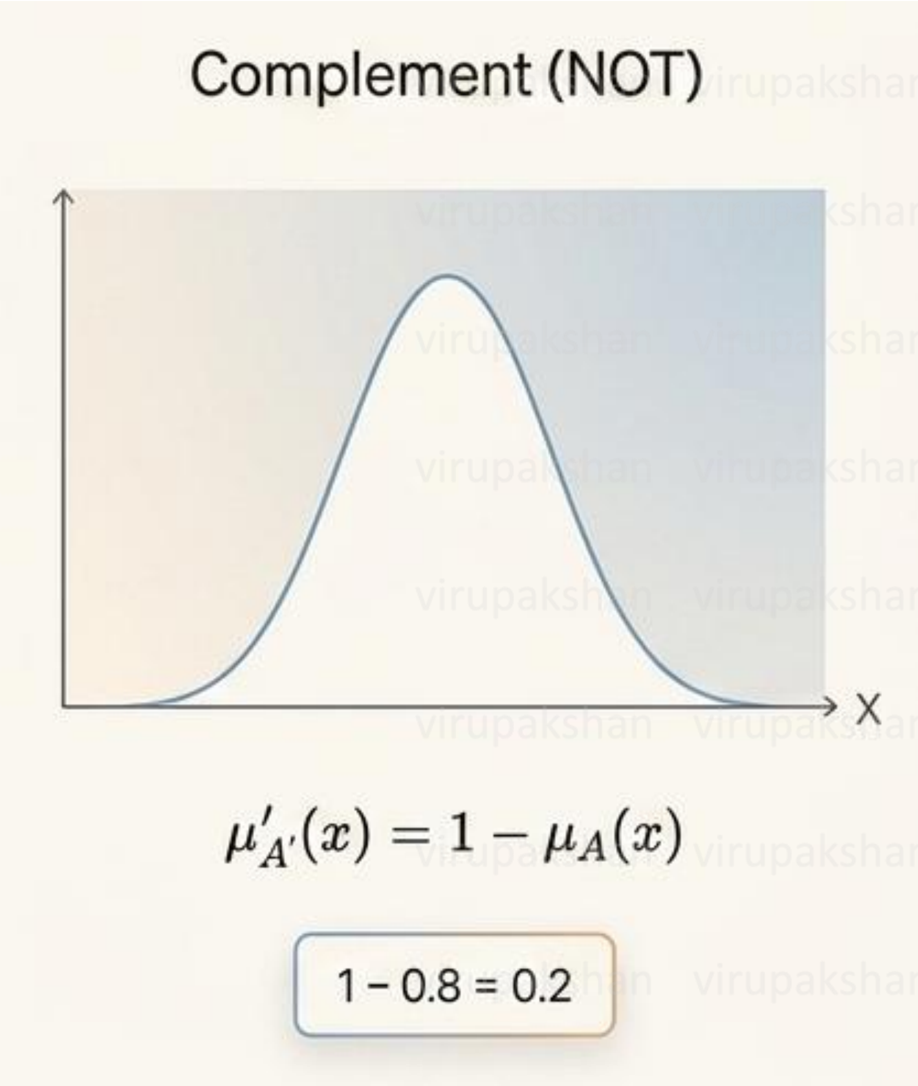
# 1.7.2b Fuzzy Set Operations: **Union (A U B)**




| **x** | $\mu_A(x)$ | $\mu_B(x)$ |
|---|---|---|
| x1 | 0.2 | 0.6 |
| x2 | 0.7 | 0.4 |
| x3 | 1.0 | 0.8 |

| **x** | **Union** |
|---|---|
| x1 | max(0.2, 0.6) = 0.6 |
| x2 | max(0.7, 0.4) = 0.7 |
| x3 | max(1.0, 0.8) = 1.0 |

# 1.7.2c Fuzzy Set Operations: Complement ( $A'$ )




| x | $\mu_A(x)$ | $\mu_B(x)$ |
|---|---|---|
| x1 | 0.2 | 0.6 |
| x2 | 0.7 | 0.4 |
| x3 | 1.0 | 0.8 |

| x | Complement of A |
|---|---|
| x1 | 0.8 |
| x2 | 0.3 |
| x3 | 0.0 |

# 1.7.2d Fuzzy Set Operations

| x | $\mu_A(x)$ | $\mu_B(x)$ |
|---|---|---|
| x1 | 0.2 | 0.6 |
| x2 | 0.7 | 0.4 |
| x3 | 1.0 | 0.8 |

| x | Intersection |
|---|---|
| x1 | min(0.2, 0.6) = 0.2 |
| x2 | min(0.7, 0.4) = 0.4 |
| x3 | min(1.0, 0.8) = 0.8 |

| x | Union |
|---|---|
| x1 | max(0.2, 0.6) = 0.6 |
| x2 | max(0.7, 0.4) = 0.7 |
| x3 | max(1.0, 0.8) = 1.0 |

| x | Complement of A |
|---|---|
| x1 | 0.8 |
| x2 | 0.3 |
| x3 | 0.0 |

# 1.7.3Fuzzy Set Operations

python

```python
A = [0.2, 0.7, 1.0]
B = [0.6, 0.4, 0.8]

union = [max(a, b) for a, b in zip(A, B)]
intersection = [min(a, b) for a, b in zip(A, B)]
complement_A = [1 - a for a in A]
difference = [min(a, 1 - b) for a, b in zip(A, B)]

print("Union:", union)
print("Intersection:", intersection)
print("Complement of A:", complement_A)
print("A - B:", difference)
```

# 1.7.4q Fuzzy Set Operations



**Question:** Given $A = \{0.3/x_1, 0.8/x_2\}$ and $B = \{0.6/x_1, 0.4/x_2\}$, find $A \cup B$ and $A \cap B$.

# 1.7.4a Fuzzy Set Operations



**Question:** Given $A = \{0.3/x_1, 0.8/x_2\}$ and $B = \{0.6/x_1, 0.4/x_2\}$, find $A \cup B$ and $A \cap B$.

**Answer:**

Union:

$$A \cup B = \{0.6/x_1, 0.8/x_2\}$$

Intersection:

$$A \cap B = \{0.3/x_1, 0.4/x_2\}$$

# 1.8.1 Properties of Fuzzy Sets

**Important Properties:** Let $A$, $B$, and $C$ be fuzzy sets.

### 1. Commutative Law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

### 2. Associative Law

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

### 3. Distributive Law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

### 4. Idempotent Law

$$A \cup A = A$$

$$A \cap A = A$$

### 5. Involution Law

$$(A')' = A$$

### 6. De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

# 1.8.2 Properties of Fuzzy Sets: **Worked Example: De Morgan's Law**

**Question:** Let

| x | A | B |
|---|---|---|
| x1 | 0.3 | 0.7 |
| x2 | 0.8 | 0.4 |

Find: $(A \cup B)'$ and $A' \cap B'$

**Answer:** For x1:

$$A \cup B = \max(0.3, 0.7) = 0.7$$

$$(A \cup B)' = 1 - 0.7 = 0.3$$

Also: $$A' = 0.7, \quad B' = 0.3$$

$$A' \cap B' = \min(0.7, 0.3) = 0.3$$

**Both** $(A \cup B)'$ and $A' \cap B'$ **are equal.**

# 1.8.3 Properties of Fuzzy Sets

python

```python
A = [0.3, 0.8]
B = [0.7, 0.4]

lhs = [1 - max(a, b) for a, b in zip(A, B)]
rhs = [min(1 - a, 1 - b) for a, b in zip(A, B)]

print("LHS:", lhs)
print("RHS:", rhs)
print("De Morgan law satisfied:", lhs == rhs)
```

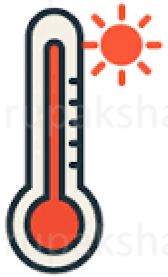
# 1.9.1 Noninteractive Fuzzy Sets

Two fuzzy sets are called **noninteractive** when the membership of one variable does not affect the membership of another variable.

In simple language, the variables are independent in a fuzzy sense.

For example:

- Temperature is hot.
- Humidity is high.



If temperature and humidity are considered separately, their fuzzy sets may be treated as **noninteractive**.

# 1.9.2 Noninteractive Fuzzy Sets

## Joint Membership

For noninteractive fuzzy sets $A$ and $B$, the joint membership is often calculated using:

$$\mu_{A\times B}(x,y) = \min[\mu_A(x), \mu_B(y)]$$

or sometimes using product:

$$\mu_{A\times B}(x,y) = \mu_A(x)\mu_B(y)$$

**Example:**

Let: $\mu_{\text{Hot}}(30) = 0.7$ and

$\mu_{\text{Humid}}(80) = 0.6$

Using **min operation**:

$$\mu_{\text{Hot and Humid}}(30, 80)$$

$$= \min(0.7, 0.6) = 0.6$$

Using **product operation**:

$$\mu_{\text{Hot and Humid}}(30, 80)$$

$$= 0.7 \times 0.6 = 0.42$$

# 1.9.3 Noninteractive Fuzzy Sets



Temperature fuzzy set    Humidity fuzzy set

↓    ↓

Combined using min or product

↓

Joint fuzzy set

$$\mu_{A\times B}(x,y) = \min[\mu_A(x), \mu_B(y)]$$

OR

$$\mu_{A\times B}(x,y) = \mu_A(x)\mu_B(y)$$

# 1.9.4 Noninteractive Fuzzy Sets

python

```python
hot = 0.7
humid = 0.6

joint_min = min(hot, humid)
joint_product = hot * humid

print("Joint membership using min:", joint_min)
print("Joint membership using product:", joint_product)
```

# 1.9.5 Noninteractive Fuzzy Sets

**Question:** What are noninteractive fuzzy sets? Give an example.

**Answer:**

Noninteractive fuzzy sets are fuzzy sets where membership in one set does not influence membership in another set.

For **example**, temperature being hot and humidity being high may be treated as noninteractive if they are considered independent variables.

# 1.10.1 Alternative Fuzzy Set Operations



Standard fuzzy operations use:

- Union = max
- Intersection = min
- Complement = 1 minus membership

But other operations are also possible. These are called **alternative fuzzy set operations.**

# 1.10.2 Alternative Fuzzy Set Operations: Common Alternative Operations



## 1. Algebraic Product

Used for fuzzy intersection:

$$\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$$

## 2. Algebraic Sum

Used for fuzzy union:

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

## 3. Bounded Difference

$$\mu_{A\cap B}(x) = \max[0, \mu_A(x) + \mu_B(x) - 1]$$

## 4. Bounded Sum

$$\mu_{A\cup B}(x) = \min[1, \mu_A(x) + \mu_B(x)]$$

# 1.10.3 Alternative Fuzzy Set Operations: Worked Example

Let: $\mu_A(x) = 0.6, \quad \mu_B(x) = 0.7$

## 1. Algebraic Product

$$\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$$

0.6×0.7 =**0.42**

## 3. Bounded Difference

$$\mu_{A\cap B}(x) = \max[0, \mu_A(x) + \mu_B(x) - 1]$$

max(0,0.6+0.7−1) =**0.3**

## 2. Algebraic Sum:

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

0.6+0.7−(0.6)(0.7)

= 1.3−0.42

=**0.88**

## 4. Bounded Sum

$$\mu_{A\cup B}(x) = \min[1, \mu_A(x) + \mu_B(x)]$$

min(1, 0.6+0.7) =**1**

# 1.10.4 Alternative Fuzzy Set Operations

python

```python
a = 0.6
b = 0.7

algebraic_product = a * b
algebraic_sum = a + b - a * b
bounded_sum = min(1, a + b)
bounded_difference = max(0, a + b - 1)

print("Algebraic product:", algebraic_product)
print("Algebraic sum:", algebraic_sum)
print("Bounded sum:", bounded_sum)
print("Bounded difference:", bounded_difference)
```

# 1.10.5 Alternative Fuzzy Set Operations

- Alternative operations provide different ways to combine fuzzy sets.
- Algebraic product is used for AND.
- Algebraic sum is used for OR.
- Bounded sum limits the result to 1.

```
Standard operations:
AND → min
OR  → max

Alternative operations:
AND → product or bounded difference
OR  → algebraic sum or bounded sum
```

# 1.10.6q Alternative Fuzzy Set Operations



**Question:** For $\mu_A(x)$ =0.5 and $\mu_B(x)$ =0.8, calculate algebraic product and algebraic sum.

# 1.10.6a Alternative Fuzzy Set Operations

**Question:** For $\mu_A(x)$ =0.5 and $\mu_B(x)$ =0.8, calculate algebraic product and algebraic sum.

**Answer:**

**Algebraic product:** $\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x)$

= 0.5×0.8 =0.4

**Algebraic sum:** $\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$

= 0.5+0.8 − (0.5)☒(0.8)

=1.3−0.4 =0.9

# 1.11.1 Crisp Relations

A relation shows a connection between elements of two sets.

A **crisp relation** either exists or does not exist.

So its value is either 0 or 1.

**Definition:**

**Let**: $A=\{a1,a2\}$ and $B=\{b1,b2\}$

A **crisp relation** $\boldsymbol{R}$ from $A$ to $B$ **is a subset of:** $\boldsymbol{A\times B}$

where: $A\times B = \{(a,b)|a\in A, b\in B\}$

# 1.11.2 Crisp Relations



**Matrix Representation**

If relation exists, write 1.

If relation does not exist, write 0.

Example: $$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# 1.11.3 Crisp Relations

**Worked Example**

Let: $A$={1,2,3}

Relation:

x is less than y

on set $A$.

**Find relation matrix.**

All pairs: (1,1), (1,2), (1,3)

(2,1), (2,2), (2,3)

(3,1), (3,2), (3,3)

Pairs where $x < y$:

(1,2), (1,3), (2,3)

Matrix:

| **R** | **1** | **2** | **3** |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 2 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 |

# 1.11.4 Crisp Relations

python

```python
A = [1, 2, 3]

R = []
for x in A:
    row = []
    for y in A:
        row.append(1 if x < y else 0)
    R.append(row)

for row in R:
    print(row)
```

# 1.11.5 Crisp Relations

**Question:** Define crisp relation with example.

**Answer:**
A crisp relation between two sets is a subset of their Cartesian product.
It shows whether a relationship exists or not using values 0 and 1.

For example, on $A$ ={1,2,3}, the relation “less than” contains ordered pairs (1,2),(1,3),(2,3).

# 1.12.1 Fuzzy Relations

A fuzzy relation is a relation where each pair has a membership value between 0 and 1.

It tells the strength of relationship between two elements.

**Definition**

A fuzzy relation $R$ from set $X$ to set $Y$ is a fuzzy set on $X \times Y$.

$$R = \{ ( (x,y), \mu_R(x,y) ) \mid x \in X, y \in Y \}$$

where:

$$0 \leq \mu_R(x,y) \leq 1$$

# 1.12.2 Fuzzy Relations

## Matrix Form

Example: $R = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 1.0 \end{bmatrix}$

This means:

- Relation between first row element and first column element is 0.2.
- Relation between second row element and second column element is 1.0.

# 1.12.3 Fuzzy Relations

**Worked Example**

Let:

```
Students = {S1, S2}

Subjects = {Math, Physics}
```

Fuzzy relation = "interest in subject"

| Student | Math | Physics |
|---|---|---|
| S1 | 0.9 | 0.6 |
| S2 | 0.4 | 0.8 |

Interpretation:

- S1 has 0.9 interest in Math.
- S2 has 0.8 interest in Physics.

# 1.12.4 Fuzzy Relations

python

```python
students = ["S1", "S2"]
subjects = ["Math", "Physics"]

R = [
    [0.9, 0.6],
    [0.4, 0.8]
]

for i, student in enumerate(students):
    for j, subject in enumerate(subjects):
        print(student, "interest in", subject, "=", R[i][j])
```

# 1.12.5 Fuzzy Relations

**Question:** What is a fuzzy relation? Give one example.

**Answer:**

A fuzzy relation is a fuzzy set defined on the Cartesian product of two sets. It assigns a membership value between 0 and 1 to each ordered pair.

For example, the relation “student interest in subject” may assign 0.9 to student S1 and Math, meaning S1 has high interest in Math.

# 1.12.6 Fuzzy Relations



## Quick Revision Points

- Fuzzy relations show degree of relationship.
- Values range from 0 to 1.
- Fuzzy relations are represented by matrices.
- They are useful in decision systems, classification, and pattern recognition.

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# LECTURE 02

## 21CSE451T

## FUZZY LOGIC AND ITS APPLICATIONS

Doubts?

THANK YOU!

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